

COMBINATORIAL GELFAND MODELS FOR SEMISIMPLE DIAGRAM ALGEBRAS

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ABSTRACT. We construct combinatorial (involutory) Gelfand models for the following diagram algebras in the case when they are semi-simple: Brauer algebra, its partial analogue, walled Brauer algebra, its partial analogue, Temperley-Lieb algebra, its partial analogue, walled Temperley-Lieb algebra, its partial analogue, partition algebra and its Temperley-Lieb analogue.

1. INTRODUCTION AND DESCRIPTION OF THE RESULT

Given an associative algebra A , a *Gelfand model* for A is an A -module which is isomorphic to the multiplicity-free direct sum of all simple A -modules, see [BGG]. If A is finite dimensional and semi-simple, then each Gelfand model for A is an additive generator of the category $A\text{-mod}$ and hence completely determines the representation theory of A in some sense.

Surprisingly enough, it turns out that in many cases there exists a relatively “easy” and “elementary” Gelfand model in comparison to, say, explicit description of all simple A -modules. One of the best examples is the combinatorial Gelfand model for complex representations of the symmetric group S_n defined as follows (see [APR1, IRS, KV]): Let \mathcal{I}_n be the set of involutions in S_n , that is the set of all elements $s \in S_n$ such that $s^2 = e$, where e is the identity element. For $\pi \in S_n$ and $s \in \mathcal{I}_n$ let $i(\pi, s)$ denote the number of pairs (i, j) , where $1 \leq i < j \leq n$, such that $s(i) = j$ and $\pi(i) > \pi(j)$. Then the formal vector space $\mathbb{C}[\mathcal{I}_n]$ with basis $\{v_s : s \in \mathcal{I}_n\}$ gets the structure of a Gelfand model for S_n by defining the action of S_n on $\mathbb{C}[\mathcal{I}_n]$ via

$$(1) \quad \pi \cdot v_s := (-1)^{i(\pi, s)} v_{\pi s \pi^{-1}} \quad \text{for all } \pi \in S_n, s \in \mathcal{I}_n.$$

The simplicity of this definition should be compared with the complexity of the classical description of the individual constituents of this module, that is the classical Specht (i.e. simple) S_n -modules over \mathbb{C} , see e.g. [Sa, Chapter 2]. As this Gelfand model is constructed in purely combinatorial terms on the set of involutions, it is sometimes also called an *involutory* or *combinatorial* Gelfand model. Similar models can be defined for many other finite groups, in particular, for all classical Weyl groups, see [APR2, Ar, ABi, ABr, Ca, CF1, CF2, GO] and references therein.

The paper [KM] makes a step beyond the group theory and constructs Gelfand models for various semigroup algebras, in particular, for semigroup algebras of inverse semigroups in which all maximal subgroups are isomorphic to direct sums of symmetric groups. This, of course, admits a straightforward generalization to inverse semigroups for which Gelfand models are known for all maximal subgroups. The goal of the present note is to make another step and construct Gelfand models for various classes of diagram algebras in the semi-simple case.

One of these diagram algebras, for which we construct a Gelfand model, is the classical Brauer algebra from [Br], which also seems to be one of the oldest diagram algebras. The other algebras are: the partial analogue of the Brauer algebra, [Maz1, Maz2, MM2]; the walled Brauer algebra, [Tu, Ko], and its partial analogue; the

(walled) Temperley-Lieb algebra, [TL], and its partial analogue, [BH, MM2]; the partition algebra, [Jo, Mar1], and its Temperley-Lieb analogue. All these algebras depend on a parameter, $\delta \in \mathbb{C}$, and they all are semi-simple for generic values of this parameter. Furthermore, all these algebras share the following properties:

- they have a combinatorial basis given by a set of diagrams;
- they have an involution \star defined combinatorially on this basis;
- they have a filtration by ideals for which subquotients “look like” some Morita-equivalent versions of (direct sums of) symmetric groups.

The main idea of the paper is that instead of the set of involutions one should consider the set of diagrams, which are self-dual with respect to \star . The formal vector space generated by this set admits the natural module structure given by “conjugation”, where the action on the right is defined using \star , and in which one should take into account two types of scalars, those coming from the definition of the diagram algebra, and those coming from the combinatorial Gelfand model for S_n as adjusted in [KM]. One essential difference with [KM] is that the set of \star -self-dual diagrams is much smaller than the set of all involutions in all maximal subgroups which was considered in [KM] (however, for inverse semigroups, that is the setup of [KM], these two sets coincide). Our approach heavily exploits the combinatorics of the classical representation theory of finite semigroups, see [GM, GMS].

The main result of the paper, which gives an explicit combinatorial Gelfand model for all diagram algebras mentioned above in the semi-simple case, is formulated and proved in Section 3 after preliminaries on diagram algebras and their representations that are collected in Section 2. For the Brauer algebra this Gelfand model was considered in [Sch].

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2. CLASSICAL DIAGRAM ALGEBRAS

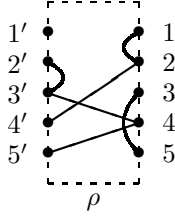
2.1. Partitions and diagrams. In this paper we work over \mathbb{C} . Fix a positive integer n and consider the sets $\underline{n} := \{1, 2, \dots, n\}$, $\underline{n}' := \{1', 2', \dots, n'\}$ and $\mathbf{n} := \underline{n} \cup \underline{n}'$ (the union is automatically disjoint). Denote by \mathbf{P}_n the set of all equivalence relations on \mathbf{n} , also known as *partitions* of \mathbf{n} . For $\rho \in \mathbf{P}_n$ an equivalence class of the equivalence relation ρ is usually called a *part* of ρ (which corresponds to viewing ρ as a partition). Define a binary operation \circ on \mathbf{P}_n , called *composition*, as follows: given $\tau, \rho \in \mathbf{P}_n$ define $\tau \circ \rho \in \mathbf{P}_n$ via the following procedure:

- First consider τ and ρ as partitions of disjoint sets
 $\{1_\tau, 2_\tau, \dots, n_\tau, 1'_\tau, 2'_\tau, \dots, n'_\tau\}$ and $\{1_\rho, 2_\rho, \dots, n_\rho, 1'_\rho, 2'_\rho, \dots, n'_\rho\}$,
 respectively.
- Identify i_τ with i'_ρ for all $i \in \underline{n}$.
- Define χ as the minimal equivalence relation on the set
 $\{1_\rho, 2_\rho, \dots, n_\rho, 1'_\rho = 1_\tau, 2'_\rho = 2_\tau, \dots, n'_\rho = n_\tau, 1'_\tau, 2'_\tau, \dots, n'_\tau\}$
 which contains both τ and ρ .
- Define $\tau \circ \rho$ as the restriction of χ to $\{1_\rho, 2_\rho, \dots, n_\rho, 1'_\tau, 2'_\tau, \dots, n'_\tau\}$ and identify the latter set with \mathbf{n} by removing all subscripts.

Denote also by $c(\tau, \rho)$ the number of equivalence classes of χ which are subsets of the set $\{1'_\rho = 1_\tau, 2'_\rho = 2_\tau, \dots, n'_\rho = n_\tau\}$.

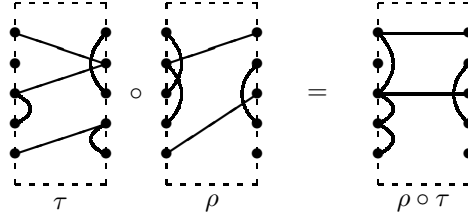
A partition $\rho \in \mathbf{P}_n$ is usually depicted as a diagram as shown in Figure 1. An example of composition of two partitions is given in Figure 2.

For $\rho \in \mathbf{P}_n$ the *rank* of ρ , denoted $\mathbf{r}(\rho)$, is the number of parts in ρ which intersect both \underline{n} and \underline{n}' non-trivially. Parts with this property are called *propagating lines*. A



$$\rho = \{1, 2, 4'\} \cup \{3, 5\} \cup \{4, 2', 3', 5'\} \cup \{1'\} \in \mathbf{P}_5$$

FIGURE 1. The diagram of a partition

FIGURE 2. Composition of partitions, $c(\tau, \rho) = 1$

part of ρ is called a *right* or *left* part if it is contained in \underline{n} or \underline{n}' , respectively. Each part of ρ is either a left or right part or a propagating line. We have $0 \leq \mathbf{r}(\rho) \leq n$.

2.2. Diagram algebras. For a fixed $\delta \in \mathbb{C}$ (which is always assumed to be nonzero in this paper), the formal linear span \mathbf{P}_n of \mathbf{P}_n has the natural structure of an associative algebra defined for the basis elements via $\tau\rho := \delta^{c(\tau, \rho)}\tau \circ \rho$ and extended to the whole of \mathbf{P}_n by bilinearity. This algebra $\mathbf{P}_n = \mathbf{P}_n(\delta)$ is called the *partition algebra*, see [Jo, Mar1]. Other diagram algebras are subalgebras of \mathbf{P}_n constructed as linear spans of certain sets of diagrams below.

Let \mathbf{B}_n and \mathcal{PB}_n denote the sets of all $\rho \in \mathbf{P}_n$ such that all parts of ρ have cardinality two or at most two, respectively. Let \mathbf{B}_n and \mathcal{PB}_n be the linear spans of \mathbf{B}_n and \mathcal{PB}_n inside \mathbf{P}_n , respectively. It is easy to see that both \mathbf{B}_n and \mathcal{PB}_n are subalgebras of \mathbf{P}_n called the *Brauer algebra* (see [Br]) and the *partial Brauer algebra* (see [Maz1, Maz2, MM2]), respectively. Note that, if $\rho \in \mathbf{B}_n$, then n and $\mathbf{r}(\rho)$ have the same parity.

Next, let \mathcal{QB}_n , \mathcal{QPB}_n and \mathcal{QP}_n denote the sets of planar diagrams in \mathbf{B}_n , \mathcal{PB}_n and \mathbf{P}_n , respectively. The linear spans \mathbf{TL}_n , \mathcal{PTL}_n and \mathbf{PTL}_n of \mathcal{QB}_n , \mathcal{QPB}_n and \mathcal{QP}_n , respectively, are also subalgebras of \mathbf{P}_n called the *Temperley-Lieb algebra* (see [TL]), *partial Temperley-Lieb algebra* (see [MM1], also called *Motzkin algebra* in [BH]) and *Temperley-Lieb-partition algebra* (see [Mar2]), respectively.

Finally, let a and b be two positive integers such that $a + b = n$. Denote by $\mathcal{W}_{a,b}\mathbf{B}_n$, $\mathcal{W}_{a,b}\mathcal{PB}_n$, $\mathcal{W}_{a,b}\mathcal{QB}_n$ and $\mathcal{W}_{a,b}\mathcal{QPB}_n$ the subsets of \mathbf{B}_n , \mathcal{PB}_n , \mathcal{QB}_n and \mathcal{QPB}_n , respectively, consisting of all diagrams which satisfy the following conditions:

- each propagating line is either contained in \mathbf{a} or in $\mathbf{n} \setminus \mathbf{a}$;
- each non-singleton right part intersects both \underline{a} and $\underline{n} \setminus \underline{a}$;
- each non-singleton left part intersects both \underline{a}' and $\underline{n}' \setminus \underline{a}'$.

The linear spans $\mathcal{W}_{a,b}\mathbf{B}_n$, $\mathcal{W}_{a,b}\mathcal{PB}_n$, $\mathcal{W}_{a,b}\mathbf{TL}_n$ and $\mathcal{W}_{a,b}\mathcal{PTL}_n$ of $\mathcal{W}_{a,b}\mathbf{B}_n$, $\mathcal{W}_{a,b}\mathcal{PB}_n$, $\mathcal{W}_{a,b}\mathcal{QB}_n$ and $\mathcal{W}_{a,b}\mathcal{QPB}_n$, respectively, are subalgebras of \mathbf{P}_n called the

walled Brauer algebra (see [Tu, Ko]), its partial analogue, *walled Temperley-Lieb algebra* and its partial analogue, respectively.

The algebra \mathbf{P}_n has a natural anti-involution \star given by swapping \underline{n} and \underline{n}' . Diagrammatically this corresponds to reflection of a diagram with respect to the vertical axes. This anti-involution restricts to all subalgebras introduced above.

2.3. Specht modules. Let \mathbf{A} denote one of the algebras \mathbf{P}_n , \mathbf{B}_n , $\mathcal{P}\mathbf{B}_n$, \mathbf{TL}_n , $\mathcal{P}\mathbf{TL}_n$, \mathbf{PTL}_n , $\mathcal{W}_{a,b}\mathbf{B}_n$, $\mathcal{W}_{a,b}\mathcal{P}\mathbf{B}_n$, $\mathcal{W}_{a,b}\mathbf{TL}_n$ or $\mathcal{W}_{a,b}\mathcal{P}\mathbf{TL}_n$. Denote by \mathbf{A} the diagram basis of \mathbf{A} .

Two elements $\tau, \rho \in \mathbf{A}$ are called

- *left equivalent*, denoted $\tau \sim_L \rho$, provided that their restrictions to \underline{n} coincide and they have the same right parts;
- *right equivalent*, denoted $\tau \sim_R \rho$, provided that their restrictions to \underline{n}' coincide and they have the same left parts.

We have $\tau \sim_L \rho$ if and only if $\mathbf{A}\tau = \mathbf{A}\rho$. Similarly, $\tau \sim_R \rho$ if and only if $\tau\mathbf{A} = \rho\mathbf{A}$. For $\tau, \rho \in \mathbf{A}$ write $\tau \sim_J \rho$ provided that $\mathbf{A}\tau\mathbf{A} = \mathbf{A}\rho\mathbf{A}$.

For $\rho \in \mathcal{W}_{a,b}\mathcal{P}\mathbf{B}_n$ denote by $\mathbf{r}_a(\rho)$ the number of propagating lines of ρ contained in \mathbf{n}_a , the so-called *a-rank* of ρ . The following lemma is straightforward and left to the reader:

Lemma 1.

- (a) If $\mathbf{A} \neq \mathcal{W}_{a,b}\mathcal{P}\mathbf{B}_n, \mathcal{W}_{a,b}\mathcal{P}\mathbf{TL}_n$, then $\tau \sim_J \rho$ is equivalent to $\mathbf{r}(\tau) = \mathbf{r}(\rho)$.
- (b) If $\mathbf{A} = \mathcal{W}_{a,b}\mathcal{P}\mathbf{B}_n$ or $\mathbf{A} = \mathcal{W}_{a,b}\mathcal{P}\mathbf{TL}_n$, then $\tau \sim_J \rho$ is equivalent to $\mathbf{r}(\tau) = \mathbf{r}(\rho)$ and $\mathbf{r}_a(\tau) = \mathbf{r}_a(\rho)$.

Clearly, both relations \sim_L and \sim_R are subsets of the relation \sim_J . Furthermore, \sim_J is the minimal equivalence relation containing both \sim_L and \sim_R , see e.g. [GM, Section 4.4]. Denote by $\text{spec}(\mathbf{A})$ the set of J -equivalence classes of diagrams.

Let \mathcal{L} be a left equivalence class in \mathbf{A} and let $\mathcal{J} \in \text{spec}(\mathbf{A})$ be the J -class containing \mathcal{L} . Then the formal linear span $\mathbb{C}[\mathcal{L}]$, considered as a subquotient of the left regular \mathbf{A} -module, inherits the natural structure of an \mathbf{A} -module given, for $\tau \in \mathbf{A}$ and $\rho \in \mathcal{L}$, by

$$\tau \cdot \rho := \begin{cases} \delta^{c(\tau, \rho)} \tau \circ \rho, & \tau \circ \rho \in \mathcal{L}; \\ 0, & \text{otherwise.} \end{cases}$$

The module $\mathbb{C}[\mathcal{L}]$ does not depend, up to isomorphism, on the choice of $\mathcal{L} \subset \mathcal{J}$ (this is similar to e.g. [MM2, Proposition 9]).

Note that all elements of \mathcal{L} have both the same rank and the same a -rank (the latter in the case of walled algebras), which we denote by $\mathbf{r}(\mathcal{L})$ and $\mathbf{r}_a(\mathcal{L})$, respectively. Consider the group

$$G = G_{\mathcal{J}} := \begin{cases} S_{\mathbf{r}_a(\mathcal{L})} \oplus S_{\mathbf{r}(\mathcal{L}) - \mathbf{r}_a(\mathcal{L})}, & \mathbf{A} \in \{\mathcal{W}_{a,b}\mathbf{B}_n, \mathcal{W}_{a,b}\mathcal{P}\mathbf{B}_n\}; \\ S_{\mathbf{r}(\mathcal{L})}, & \mathbf{A} \in \{\mathbf{P}_n, \mathbf{B}_n, \mathcal{P}\mathbf{B}_n\}; \\ \{e\}, & \text{otherwise.} \end{cases}$$

There is an obvious right action of G on the \mathbf{A} -module $\mathbb{C}[\mathcal{L}]$ by automorphisms which permute the right ends of the propagating lines. This makes $\mathbb{C}[\mathcal{L}]$ into an \mathbf{A} - $\mathbb{C}[G]$ -bimodule which is free over $\mathbb{C}[G]$. The rank of the latter free module is the number $q_{\mathcal{J}}$ of right equivalence classes in \mathcal{J} (which is also equal to the number of left equivalence classes because of \star). If N is a simple G -module, we can consider the corresponding *Specht* \mathbf{A} -module $\Delta_{\mathcal{L}}(N) := \mathbb{C}[\mathcal{L}] \otimes_{\mathbb{C}[G]} N$. Note that such simple modules N are described by

- pairs (λ_1, λ_2) of partitions $\lambda_1 \vdash \mathbf{r}_a(\mathcal{L})$ and $\lambda_2 \vdash \mathbf{r}(\mathcal{L}) - \mathbf{r}_a(\mathcal{L})$ in the case $\mathbf{A} \in \{\mathcal{W}_{a,b}\mathbf{B}_n, \mathcal{W}_{a,b}\mathcal{P}\mathbf{B}_n\}$;

- partitions $\lambda \vdash \mathbf{r}(\mathcal{L})$ in the case $\mathbf{A} \in \{\mathbf{P}_n, \mathbf{B}_n, \mathcal{P}\mathbf{B}_n\}$;
- the trivial partition $\lambda \vdash 1$ in all Temperley-Lieb cases.

We have $\dim \Delta_{\mathcal{L}}(N) = q_{\mathcal{J}} \dim N$. For each $\mathcal{J} \in \text{spec}(\mathbf{A})$ fix some left equivalence class $\mathcal{L}_{\mathcal{J}}$ contained in \mathcal{J} .

Theorem 2.

- (a) *The module $\Delta_{\mathcal{L}_{\mathcal{J}}}(N)$ is simple for all but finitely many values of δ .*
- (b) *\mathbf{A} is semi-simple if and only if all $\Delta_{\mathcal{L}_{\mathcal{J}}}(N)$ are simple.*
- (c) *If \mathbf{A} is semi-simple, then the set of all $\Delta_{\mathcal{L}_{\mathcal{J}}}(N)$, where $\mathcal{J} \in \text{spec}(\mathbf{A})$ and N runs through the set of isoclasses of simple $G_{\mathcal{J}}$ -modules, forms a complete and irredundant set of pairwise non-isomorphic simple \mathbf{A} -modules.*

2.4. Sketch of the proof of Theorem 2. In most cases, Theorem 2 can be found, sometimes disguised, in the existing literature. In other cases it is proved using the same arguments as in the known cases. Here we sketch the standard approach to its proof.

The fact that \mathbf{A} is a union of left equivalence classes means that the left regular representation of \mathbf{A} has a filtration whose subquotients are isomorphic to Specht modules. The endomorphism algebra of $\Delta_{\mathcal{L}}(N)$ is isomorphic to the endomorphism algebra of the G -module N which, in turn, is isomorphic to \mathbb{C} . This implies that \mathbf{A} is quasi-hereditary (note that we assume $\delta \neq 0$), with Specht modules being the standard modules for the quasi-hereditary structure. Moreover, \mathbf{A} has a simple preserving duality induced by \star . Now from the BGG reciprocity it follows that \mathbf{A} is semi-simple if and only if all $\Delta_{\mathcal{L}}(N)$ are simple (which is claim (b)). Since Specht modules are standard modules, claim (b) implies claim (c).

It remains to address claim (a). For the Temperley-Lieb algebra, partition algebra and its Temperley-Lieb analogue this claim can be found in [Mar2]. For the Brauer algebra, see [Ru] and references therein. For the walled Brauer algebra, see [Ha, Ni], and the same argument works for the walled Temperley-Lieb algebra. For the partial Brauer algebra, see [MM2]. The argument from [MM2] restricts to all other partial algebras as well (i.e. for partial Temperley-Lieb algebras, partial walled Brauer algebras and partial walled Temperley-Lieb algebras). This completes the proof.

3. THE MAIN RESULT

3.1. The model. Let \mathbf{A} denote one of the algebras $\mathbf{P}_n, \mathbf{B}_n, \mathcal{P}\mathbf{B}_n, \mathbf{TL}_n, \mathcal{P}\mathbf{TL}_n, \mathbf{PTL}_n, \mathcal{W}_{a,b}\mathbf{B}_n, \mathcal{W}_{a,b}\mathcal{P}\mathbf{B}_n, \mathcal{W}_{a,b}\mathbf{TL}_n$ and $\mathcal{W}_{a,b}\mathcal{P}\mathbf{TL}_n$. Denote by \mathbf{A} the diagram basis of \mathbf{A} and let \mathcal{I} denote the set of all \star -self-dual elements in \mathbf{A} .

Let $\tau \in \mathbf{A}$ be a diagram of rank $k \in \{0, 1, \dots, n\}$. We associate to τ an element $\pi_{\tau} \in S_k$ in the following way: Let A_1, A_2, \dots, A_k be the list of propagating lines in τ . For $i \in \{1, 2, \dots, k\}$ let B_i and C_i denote the intersections of A_i with \underline{n} and \underline{n}' , respectively. We assume that for all $1 \leq i < j \leq k$ we have $\min\{s \in B_i\} < \min\{s \in B_j\}$. We now define $\pi_{\tau} \in S_k$ as the unique permutation such that for all $1 \leq i < j \leq k$ we have $\min\{s \in C_{\pi_{\tau}(i)}\} < \min\{s \in C_{\pi_{\tau}(j)}\}$. For example, for the element ρ in Figure 1 we have $k = 2$, $A_1 = \{1, 2, 4'\}$, $A_2 = \{4, 2', 3', 5'\}$, $B_1 = \{1, 2\}$, $B_2 = \{4\}$, $C_1 = \{4'\}$, $C_2 = \{2', 3', 5'\}$ and hence $\pi_{\rho} \in S_2$ is the transposition swapping 1 and 2. For the element τ in Figure 2 we similarly get that $\pi_{\tau} \in S_2$ is the identity map. For $\iota \in \mathcal{I}$ we note that $\{i, j'\}$ belongs to a propagating line if and only if $\{i', j\}$ does. This implies that for $\iota \in \mathcal{I}$ the element $\pi_{\iota} \in S_2$ is an involution. If \mathbf{A} is one of the walled algebras, then, by construction, the element π_{τ} belongs to $S_{\mathbf{r}_a(\tau)} \oplus S_{k-\mathbf{r}_a(\tau)}$.

Consider the formal span $\mathbb{C}[\mathcal{I}]$ with basis $\{v_\iota : \iota \in \mathcal{I}\}$ and for $\tau \in \mathbf{A}$ and $\iota \in \mathcal{I}$ set

$$(2) \quad \tau \cdot v_\iota := \begin{cases} (-1)^{i(\pi_{\tau \circ \iota} \pi_\iota, \pi_\iota)} \delta^{c(\tau, \iota)} v_{\tau \circ \iota \circ \tau^*}, & \mathbf{r}(\tau \circ \iota) = \mathbf{r}(\iota); \\ 0, & \text{otherwise.} \end{cases}$$

Here $i(\pi_{\tau \circ \iota} \pi_\iota, \pi_\iota)$ is defined as in Section 1. Note that $\mathbf{r}(\tau \circ \iota) = \mathbf{r}(\iota)$ also implies $\mathbf{r}(\tau \circ \iota \circ \tau^*) = \mathbf{r}(\iota)$ by symmetry. The explanation for the “complicated-looking” element $\pi_{\tau \circ \iota} \pi_\iota$ is the following: for the formula to make sense we need an element “just like τ but of the same rank as ι ” which would act on propagating lines of ι , the easiest way to produce such an element is to multiply τ with ι , however, ι is not idempotent and hence one has to compensate by multiplying with $\pi_\iota = \pi_\iota^{-1}$ (recall that π_ι is an involution). We can now state our main result:

Theorem 3. *If \mathbf{A} is semi-simple, then formula (2) defines on $\mathbb{C}[\mathcal{I}]$ the structure of a Gelfand model for \mathbf{A} .*

3.2. Proof of Theorem 3. Theorem 3 follows from the following two lemmata:

Lemma 4. *Formula (2) defines on $\mathbb{C}[\mathcal{I}]$ the structure of an \mathbf{A} -module.*

Proof. Let $\tau, \rho \in \mathbf{A}$ and $\iota \in \mathcal{I}$. We have to show that $(\tau\rho) \cdot v_\iota = \tau \cdot (\rho \cdot v_\iota)$. Using associativity of \circ and involutivity of \star we have

$$(\tau \circ \rho) \circ \iota \circ (\tau \circ \rho)^* = \tau \circ (\rho \circ \iota \circ \rho^*) \circ \tau^* =: \alpha.$$

This implies that both $(\tau\rho) \cdot v_\iota$ and $\tau \cdot (\rho \cdot v_\iota)$ are scalar multiples of the same basis vector.

Now let us check that the corresponding scalars coincide. Both scalars are non-zero if and only if $\mathbf{r}(\alpha) = \mathbf{r}(\iota)$, so from now on we assume the latter equality. Both scalars are powers of δ up to sign. The powers of δ agree because of the fact that the correction term $\delta^{c(\tau, \iota)}$ does define an associative algebra structure on \mathbf{A} . So, the only thing we are left to check is the fact that

$$(3) \quad (-1)^{i(\pi_{\tau \circ \rho \circ \iota} \pi_\iota, \pi_\iota)} = (-1)^{i(\pi_{\tau \circ (\rho \circ \iota \circ \rho^*)} \pi_{(\rho \circ \iota \circ \rho^*)}, \pi_{(\rho \circ \iota \circ \rho^*)})} (-1)^{i(\pi_{\rho \circ \iota} \pi_\iota, \pi_\iota)}.$$

The fact that (1) defines a module structure over the symmetric group reduces verification of (3) to the following identity:

$$(4) \quad \pi_{\tau \circ \rho \circ \iota} \pi_\iota = \pi_{\tau \circ (\rho \circ \iota \circ \rho^*)} \pi_{(\rho \circ \iota \circ \rho^*)} \pi_{\rho \circ \iota} \pi_\iota.$$

Let $A_1, A_2, \dots, A_{\mathbf{r}(\iota)}$ denote intersections of propagating lines of ι with \underline{n} ordered such that for all $1 \leq i < j \leq \mathbf{r}(\iota)$ we have $\min\{s \in A_i\} < \min\{s \in A_j\}$. Let $B_1, B_2, \dots, B_{\mathbf{r}(\iota)}$ denote intersections of propagating lines of $\rho \circ \iota$ with \underline{n}' ordered in the similar way and, finally, let $C_1, C_2, \dots, C_{\mathbf{r}(\iota)}$ denote intersections of propagating lines of $\tau \circ \rho \circ \iota$ with \underline{n}' ordered in the similar way. Then, by the definition of π , the propagating line of $\tau \circ \rho \circ \iota$ containing A_i also contains $C_{\pi_{\tau \circ \rho \circ \iota}(i)}$. Similarly, the propagating line of $\rho \circ \iota$ containing A_i also contains $B_{\pi_{\rho \circ \iota}(i)}$ and, finally, the propagating line of $\tau \circ (\rho \circ \iota \circ \rho^*)$ containing the unprimed version of B_i also contains $C_{\pi_{\tau \circ (\rho \circ \iota \circ \rho^*)}(i)}$. This reduces the equality (4) to the following equality:

$$\tau \circ \rho \circ \iota \circ \iota = \tau \circ (\rho \circ \iota \circ \rho^*) \circ (\rho \circ \iota \circ \rho^*) \circ \rho \circ \iota \circ \iota.$$

This last equality follows from

$$(\rho \circ \iota \circ \rho^*) \circ (\rho \circ \iota \circ \rho^*) \circ \rho \circ \iota = \rho \circ \iota,$$

which is proved by comparing the left and right parts on both the left and the right hand sides of the equality since both these sides have the same rank and the element $(\rho \circ \iota \circ \rho^*) \circ (\rho \circ \iota \circ \rho^*)$ is idempotent (the latter is equivalent to the assertion that π sends \star -self-dual elements to involutions). This completes the proof of the lemma. \square

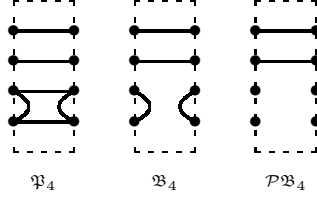


FIGURE 3. The element τ from the proof of Lemma 5 for $n = 4$ and $k = 2$

Lemma 5. *If \mathbf{A} is semi-simple, then the \mathbf{A} -module $\mathbb{C}[\mathcal{I}]$ is a multiplicity free direct sum of all Specht \mathbf{A} -modules.*

Proof. For $\mathcal{J} \in \text{spec}(\mathbf{A})$ let $\mathcal{I}_{\mathcal{J}}$ denote the set of elements of \mathcal{I} which are in \mathcal{J} . Set $k := \mathbf{r}(\mathcal{J})$. From the definition we have that $\mathbb{C}[\mathcal{I}]$ decomposes into a direct sum of $\mathbb{C}[\mathcal{I}_{\mathcal{J}}]$ 'th for $\mathcal{J} \in \text{spec}(\mathbf{A})$. So, it is enough to show that $\mathbb{C}[\mathcal{I}_{\mathcal{J}}]$ is a multiplicity free direct sum of Specht modules $\Delta_{\mathcal{L}_{\mathcal{J}}}(N)$ where N runs through the set of isomorphism classes of simple $G_{\mathcal{J}}$ -modules. Note that, by construction, the dimension of $\mathbb{C}[\mathcal{I}_{\mathcal{J}}]$ equals $q_{\mathcal{J}}$ times the number of involutions in $G_{\mathcal{J}}$ and hence agrees with the dimension of the multiplicity free direct sum of Specht modules as above. Hence if \mathbf{A} is any of the Temperley-Lieb type algebras, then the claim is obvious. Similarly in the case $k = 0, 1$ for any \mathbf{A} . So we are left with the cases $\mathbf{A} \in \{\mathbf{P}_n, \mathbf{B}_n, \mathcal{PB}_n, \mathcal{W}_{a,b}\mathbf{B}_n, \mathcal{W}_{a,b}\mathcal{PB}_n\}$ and $k > 1$.

Assume first that $\mathbf{A} = \mathbf{P}_n$. Consider the following element:

$$\tau = \{1, 1'\} \cup \{2, 2'\} \cup \cdots \cup \{k-1, (k-1)'\} \cup \{k, k+1, \dots, n, k', (k+1)', \dots, n'\}$$

(see an example in Figure 3). We have $\tau = \tau^*$ and $\tau \circ \tau = \tau = \tau \tau$. Let H denote the set of all diagrams ρ of rank k satisfying $\tau \circ \rho = \rho$ and $\rho \circ \tau = \rho$. Then $\rho_1 \circ \rho_2 = \rho_1 \rho_2$ for all $\rho_1, \rho_2 \in H$ and hence the linear span of H is isomorphic to the group algebra of the symmetric group S_k and τ is the identity element in H .

Without loss of generality we may assume that $H \subset \mathcal{L}_{\mathcal{J}}$. In this case from the definition of $\Delta_{\mathcal{L}_{\mathcal{J}}}(N)$ we may identify the image of the linear operator defined by τ acting on $\Delta_{\mathcal{L}_{\mathcal{J}}}(N)$ with N . Now, comparing (2) and (1) we see that the image of the linear operator defined by τ acting on $\mathbb{C}[\mathcal{I}_{\mathcal{J}}]$ can be identified with a Gelfand model for H . It follows that $\mathbb{C}[\mathcal{I}_{\mathcal{J}}]$ is a multiplicity free direct sum of Specht modules $\Delta_{\mathcal{L}_{\mathcal{J}}}(N)$ where N corresponds to $\lambda \vdash k$.

If $\mathbf{A} = \mathbf{B}_n$, then we consider the following diagram (see an example in Figure 3):

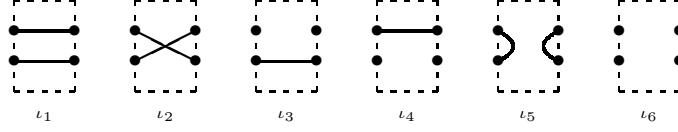
$$\tau = \{1, 1'\} \cup \cdots \cup \{k, k'\} \cup \{k+1, k+2\} \cup \{(k+1)', (k+2)'\} \cup \cdots \cup \{n-1, n\} \cup \{(n-1)', n'\}.$$

Let $l := (n-k)/2$ (note that n and k have the same parity). Let H denote the set of all diagrams ρ of rank k satisfying $\tau \circ \rho = \rho$ and $\rho \circ \tau = \rho$. Then the linear span of H is again isomorphic to the group algebra of the symmetric group S_k (acting on $\{1, 2, \dots, k\}$) by sending a diagram to δ^l times the corresponding permutation in S_k (recall that $\delta \neq 0$ by our assumptions). Now the proof is completed in the same way as in the case $\mathbf{A} = \mathbf{P}_n$.

Finally, for $\mathbf{A} = \mathcal{PB}_n$ we consider the following element:

$$\tau = \{1, 1'\} \cup \cdots \cup \{k, k'\} \cup \{k+1\} \cup \{(k+1)'\} \cup \cdots \cup \{n\} \cup \{n'\}$$

(again, see an example in Figure 3). Let H denote the set of all diagrams ρ of rank k satisfying $\tau \circ \rho = \rho$ and $\rho \circ \tau = \rho$. Then the linear span of H is again isomorphic to the group algebra of the symmetric group S_k (acting on $\{1, 2, \dots, k\}$) by sending

FIGURE 4. \star -self-dual diagrams for \mathcal{PB}_2

a diagram to δ^{n-k} times the corresponding permutation in S_k . Now the proof is completed in the same way as in the case $\mathbf{A} = \mathbf{P}_n$.

The cases $\mathbf{A} = \mathcal{W}_{a,b}\mathbf{B}_n, \mathcal{W}_{a,b}\mathcal{PB}_n$ are dealt with similarly and left to the reader. \square

Remark 6. The \mathbf{A} -module $\mathbb{C}[\mathcal{I}]$ makes also sense in the non-semisimple case in which it is just isomorphic to a direct sum of Specht \mathbf{A} -modules. The latter are the cell modules with respect to the cellular structure of \mathbf{A} . Maybe this is a sensible notion of Gelfand model for a cellular algebras?

3.3. Example. Let $\mathbf{A} = \mathcal{PB}_2$. The set \mathcal{I} consists of the six diagrams shown in Figure 4. The algebra \mathbf{A} is generated by $\alpha := \iota_2$, $\beta := \iota_3$ and $\gamma := \iota_5$. The action of these generators (denoted by π in the table below) on the basis elements of \mathcal{I} is given by:

$\pi \setminus \iota$	ι_1	ι_2	ι_3	ι_4	ι_5	ι_6
α	ι_1	$-\iota_2$	ι_4	ι_3	ι_5	ι_6
β	0	0	ι_3	0	ι_6	$\delta\iota_6$
γ	0	0	0	0	$\delta\iota_5$	$\delta\iota_5$

The module $\mathbb{C}[\mathcal{I}]$ is a direct sum of four simple \mathbf{A} -modules, namely, the linear spans of $\{\iota_1\}$, $\{\iota_2\}$, $\{\iota_3, \iota_4\}$ and $\{\iota_5, \iota_6\}$.

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